151. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If A and B play tennis together, find the probability that A will win, given that a is the probability that A will win a given point and b the probability that B will win the point.

I. Solution* by the PROPOSER.

Without reaching the score deuce, A wins [a game] only in the following cases:

AAAA, AAABA, AABAA, ABAAA, BAAAA,
AAABBA, AABBAA, ABBAAA, BBAAAA, AABABA,
ABAABA, ABABAA, BAABAA, BABAAA, BAAABA,

with the probability $a^4+4a^4b+10a^4b^2$. The number of ways in which the score deuce may first arise equals the number of permutations of A, A, A, B, B, B, which is $6!\div(3!3!)=20$. It remains to find the probability that A will win after the score deuce is reached. This happens in 2 plays in the case AA; in 4 plays in the cases ABAA, BAAA; in 6 plays in the cases ABABAA, ABBAAA, BAABAA, BABAAA; in 8 plays in the cases when either AB or BA is followed by one of the preceding 4 cases; etc. Hence the probability that A will win after deuce is

$$a^2+2a^3b+4a^4b^2+8a^5b^3+\ldots\ldots+2^na^{n+2}b^n+\ldots\ldots=\frac{a^2}{1-2ab}.$$

The total probability that A will win is therefore

$$a^4+4a^4b+10a^4b^2+\frac{20a^2}{1-2ab}a^3b^3.$$

Interchanging a and b, we obtain as the probability that B will win

$$b^4+4b^4a+10b^4a^2+\frac{20b^2}{1-2ab}b^3a^3.$$

As a check, it is seen that this sum equals 1 in view of $a+b=1$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

To win a game in the first six points A must win four straight points, or four out of five, or four out of six. The chance of this is

$$a^4+\frac{4a^4b}{1!}+\frac{4.5a^4b^2}{2!}=a^4+4a^4b+10a^4b^2.$$

The chance that B wins in the first six points is $b^4+4ab^4+10a^2b^4$. The chance that each wins three points is $(6!/3!3!)a^3b^3=20a^3b^3$. If they each win three points, then A must win two successive points to win the game. The chance that A wins two successive points is $20a^5b^3$, the chance that B wins two successive points is $20a^3b^5$. The chance that each win a point is $40a^4b^4$. After each have

*The problem was intended to read "win a game," a second proposed problem reading "win a set." The latter has been withdrawn in view of Professor Zerr's solution.

won four points, the chance that A wins two successive points is $40a^6b^4$. The chance that B wins two successive points is $40a^4b^6$, the chance that each win a point is $80a^5b^5$, and so on.

Let p=A's chance of winning the first game, and q=B's chance of winning the first game.

$$\therefore p=a^4+4a^4b+10a^4b^2+20a^5b^3+40a^6b^4+80a^7b^5+160a^8b^6+\ldots\ldots$$
$$=a^4+4a^4b+10a^4b^2+20a^5b^3(1+2ab+4a^2b^2+8a^3b^3+\ldots\ldots)$$
$$=a^4+4a^4b+10a^4b^2+\frac{20a^5b^3}{1-2ab}=\frac{a^4}{1-2ab}(1+4b+10b^2-2ab-8ab^2).$$

Similarly, $q=\dfrac{b^4}{1-2ab}(1+4a+10a^2-2ab-8a^2b)$.

The chance that A wins a set in ten games is

$$p^6+6p^6q+\frac{6.7}{2!}p^6q^2+\frac{6.7.8}{3!}p^6q^3+\frac{6.7.8.9}{4!}p^6q^4$$
$$=p^6+6p^6q+21p^2q^2+56p^6q^3+126p^6q^4.$$

The chance that B wins the set in ten games is $q^6+6q^6p+21q^6p^2+56q^6p^3+126q^6p^4$.

The chance that each wins five games is $\dfrac{10!}{5!5!}p^5q^5=252p^5q^5$.

Let P be A's and Q be B's chance of winning the first set. Then as before

$$P=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^6q^5+504p^8q^6$$
$$+1008p^9q^7+2016p^{10}q^8+\ldots\ldots$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+252p^7q^5(1+2pq$$
$$+4p^2q^2+8p^3q^3+\ldots\ldots)$$
$$=p^6+6p^6q+21p^6q^2+56p^6q^3+126p^6q^4+\frac{252\,p^7q^5}{1-2pq}$$
$$=\frac{p^6}{1-2pq}(1+6q+21q^2+56q^3+126q^4-2pq-12pq^2-42pq^3-112pq^4).$$
$$Q=\frac{q^6}{1-2pq}(1+6p+21p^2+56p^3+126p^4-2pq-12p^2q-42p^3q-112p^4q).$$

The chance that A wins three sets out of five and thus wins the match is $P_1=P^3+3P^3Q+6P^3Q=P^3(1+3Q+6Q^2)$

The chance that B wins the match is $Q_1=Q^3(1+3P+6P^2)$.

MISCELLANEOUS.

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142. Proposed by R. A. WELLS, Bellevue College, Bellevue, Neb.

Find a general expression for the value of θ such that when θ is one of the acute angles of a right triangle, the three sides of the triangle will be commensurable.

I. Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Without assuming the triangle to be right, we start with the equation

$$c^2=a^2+b^2-2ab\cos C,\ \therefore a=b\cos C\pm\sqrt{(b^2\cos^2C-b^2+c^2)}.$$